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The zero-phase Stefan problem

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Abstract

The classical one-phase Stefan problem is presented in dimensionless form with a time-varying heat-power flux boundary condition. The formulating parameters are the Stefan number, *Ste*, and a generalized form of the Biot number, *Bi*. The asymptotic solution for $Bi \to 0$ of the governing equations is of an isothermal phase change material domain, simplifying the model into a moving boundary zero-phase type problem. Exact solutions to the zero-phase model can be found for finite domains in Cartesian, cylindrical and spherical coordinates in one dimension with sign-switching boundary conditions in terms of moving boundary location, or, conversely, melting times. The model can be thought of as an analytical approximation for cases having small but finite Biot numbers. A more general expression that takes the geometry as a parameter is presented.

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Keywords: Stefan problem; Phase change modeling; Asymptotic solution; Analytical approximation; Charge–discharge; Finite domain

1. Introduction

Heat diffusion with phase change of the diffusing medium appears in a variety of natural phenomena and technological processes, and are frequently coupled with other transport phenomena. Such problems involving the solid and liquid phases are part of a broader class of moving boundary problems, or Stefan problems, named in honor of Jožef Stefan [1]. Though the underlying heat diffusion process can sometimes be considered linear, Stefan problems incorporate a non-linearity of geometrical type [2], because the moving boundary location is unknown.

Phase change materials (PCMs) are commonly used for energy storage and retrieval in thermal energy systems and electronics cooling, the thermal design of such systems involves the solution of a coupled problem. Approximate analytical models for the phase change phenomenon are useful tools in the early design stages of such systems; however, the simplest of such tools—the quasistationary approximation due to Leĭbenzon [3]—fails to accommodate initial conditions when the direction of heat transfer switches in the fixed boundary.

The solution of higher order analytical approximations, such as perturbation methods [4], quickly becomes complex and unattractive, if compared to the generality and simplicity of numerical methods [5,6]. On the other hand, analytical approximations yield, contrary to numerical methods, parametric solutions that are far more desirable in the design process of a system.

This limitation compels one to seek even simpler analytical models. Frequently, one might be interested in finding solutions for asymptotic values of the parameters [7–9]. Such approach is seemingly common in contemporary research on Stefan problems [10–12].

The present work introduces an isothermal moving boundary approach to model phase change processes. The model is formally derived from the classical one-phase Stefan problem whose boundary condition (BC) is made to depend on a generalized form of the Biot number, *Bi*, and corresponds to its asymptotic solution taking $Bi \to 0$.

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Nomenclature

Bi	Biot number, Eq. (7)
c	incompressible substance specific heat, Eq. (3)
f	any function, Eq. (15)
G	geometrical coefficient, Section 5
h	convective heat transfer coefficient, Eq. (5)
J	number of phase change interfaces in domain, Section 4.2
k	PCM thermal conductivity, Eq. (2)
L	PCM latent heat of fusion, Eq. (4)
ℓ	dimensionless domain extent, Eq. (30)
Q	heat-power flux ratio, Eq. (24)
q''	heat-power flux, Eq. (2)
R	radial phase change interface location, Eq. (34)
r	radial coordinate, Eq. (34)
s	integrand dimensionless time, Eq. (20)
Ste	Stefan number, Eq. (7)
T	temperature, Eq. (1)
t	time, Eq. (2)
X	phase change interface location, Eq. (3)
x	Cartesian coordinate, Eq. (1)
Z	dimensionless radial phase change interface location, Eq. (34)

Greek symbols

α	thermal diffusivity, Section 3.3
β	radiation heat transfer factor, Eq. (5)
ζ	dimensionless radial coordinate, Eq. (34)
Θ	dimensionless radiative heat-power flux, Eq. (14)
θ	dimensionless temperature, Eq. (6)
κ	dimensionless thermal conductivity, Eq. (8)
λ	freezing constant, Eq. (23)
ν	temperature ratio, Section 2.1
Ξ	dimensionless interface location, Eq. (6)
ξ	dimensionless Cartesian coordinate, Eq. (6)
ρ	PCM's density, Eq. (3)
Σ	dimensionless phase extent, Eq. (30)
τ	dimensionless time, Eq. (6)
ψ''	dimensionless heat-power flux, Eq. (6)

Subscripts

0	initial
f	far
i, j, k	indices
L	that of liquid phase
ℓ	left
m	melting
n	near
p	prescribed
r	right
S	that of solid phase
w	that of the wall
∞	evaluated at infinity

Superscripts

qs	quasistationary
zp	zero-phase
$'$	translated
$-$	near side of
$+$	far side of
$\star$	scale of

Owing to its simplicity, solutions to problems involving multiple interfaces in finite geometries in Cartesian, cylindrical, and spherical coordinates with time-varying, sign-switching heat-power fluxes as BCs can be easily found for the zero-phase formulation.

The isothermal or zero-phase approach for modeling phase change processes has the advantage of uncoupling otherwise coupled problems, since the PCM domain can be replaced by a fixed temperature boundary condition at the PCM's melting temperature for the rest of the system, as long as the PCM is changing phase.

The solution given by the proposed zero-phase model is exact when $Bi \to 0$ and is applicable only when the PCM is undergoing phase change. A study on lumped closed systems with heat generation that exchanges heat with a periodic temperature environment and internal PCM-filled heat sinks [13], verified that replacing the PCM domain by an isothermal BC yields results within 2% of the value obtained numerically for the Biot number equal to unity. The discrepancy between theoretical and numerical results grows rapidly, however, as the Biot number increases.

Hence, the zero-phase model can be applied as an analytical approximation for cases having small but finite Biot numbers, such as phase change in micro-channels, phase change of micro or nano-encapsulated PCMs, and some situations involving highly conductive PCMs.

As will be shown later, the present model can also be applied to problems in which heat interactions are applied directly to the phase change interface, such as melting of icebergs or non-encapsulated PCMs, radiative melting of a PCM constituted of an opaque solid and a transparent liquid, or direct contact melting.

2. The one-phase Stefan problem with time-varying heat-power flux boundary condition

Consider the classical one-phase Stefan problem, in which a semi-infinite, initially solid slab having uniform temperature distribution

$$T(x,0) = T_{\mathrm{m}}, \quad x \geqslant 0, \tag{1}$$

where T_m is the PCM melting temperature and x the Cartesian coordinate that bounds the domain at its origin, is melted by an imposed time-varying heat-power flux, $q''_w(t) \geqslant 0, \forall t \geqslant 0$, at the boundary $x = 0$:

$$q''_w(t) = -k_L \frac{\partial T}{\partial x}\bigg|_{x=0}, \tag{2}$$

where k_L is the thermal conductivity of the liquid PCM.

The unsteady temperature distribution of the liquid phase, $T(x,t)$, is governed by the heat diffusion equation on the (variable) liquid domain in the absence of advection:

$$\rho c_L \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k_L \frac{\partial T}{\partial x}\right), \quad 0 \leqslant x < X(t), \tag{3}$$

where ρ is the PCM's density (same for the solid and liquid phases), c_L is the incompressible liquid phase specific heat, which might be different from that of the solid phase, and $X(t)$ is the location of the sharp, planar liquid–solid interface, which moves according to the Stefan condition

$$\rho L \frac{dX}{dt} = -k_L \frac{\partial T}{\partial x}\bigg|_{X^-}, \tag{4}$$

where L is the PCM latent heat of fusion. The temperature at the phase change interface is the melting temperature: $T(X(t),t) = T_m$. The initial condition for Eq. (4) is $X(0) = 0$.

The time-varying heat-power flux $q''_w(t)$ that causes the phase change,

$$q''_w(t) = \begin{cases} q''_p(t), & \text{(a)} \\ h[T_\infty(t) - T_w(t)], & \text{(b)} \\ \beta[T^4_\infty(t) - T^4_w(t)], & \text{(c)} \end{cases} \tag{5}$$

models not only a prescribed amount of heat flux imposed on the boundary (a), but also a convective (b) or radiative (c) heat flux between the boundary and its surroundings.

In Eq. (5), $T_\infty(t)$ takes on the usual meaning for Newtonian and radiative heat exchanges, $T_w(t)$ is the absolute wall temperature of the phase change domain: $T_w(t) \equiv T(0,t)$, h is the convective heat transfer coefficient and β is the radiation heat transfer factor. The solidification problem is obtained by replacing k_L, c_L, and L by k_S, c_S, and $-L$, respectively [2].

2.1. Non-dimensionalization of the problem

Let $T^\star$ be L/c_L, or a scale of $|T_\infty(t) - T_w(t)|$, or $|(T^\star_\infty)^4 - (T^\star_w)^4|^{1/4}$, when option (a), (b), or (c) of Eq. (5), respectively, is prescribed, where $T^\star_\infty$ and $T^\star_w$ are scales of $T_\infty(t)$ and $T_w(t)$, respectively. Let further $q^\star$ be a scale of $|q''_p(t)|$, or $hT^\star$, or $\beta(T^\star)^4$; $t^\star$ a temporal scale of either function $q''_p(t)$ or $T_\infty(t)$; and $x^\star$ a length scale such that $x^\star = (t^\star q^\star)/(\rho L)$, so the following dimensionless variables

$$\theta = \frac{T - T_m}{T^\star}, \quad \psi = \frac{q}{q^\star}, \quad \tau = \frac{t}{t^\star}, \quad \text{and}$$

$$(\xi, \Xi) = \frac{(x, X)}{x^\star}, \tag{6}$$

and the parameters

$$Ste = \frac{c_L T^\star}{L}, \quad \text{and} \quad Bi = \frac{q^\star x^\star}{k_L T^\star}, \tag{7}$$

can be defined. With Eqs. (6) and (7), Eqs. (2)–(4) become

$$\psi''_w(\tau) = \frac{-\kappa}{Bi}\frac{\partial \theta}{\partial \xi}\bigg|_{\xi=0}, \tag{8}$$

$$Ste \frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial \xi}\left(\frac{\kappa}{Bi}\frac{\partial \theta}{\partial \xi}\right), \quad 0 \leqslant \xi < \Xi(\tau), \tag{9}$$

$$\pm \frac{d\Xi}{d\tau} = \frac{-\kappa}{Bi}\frac{\partial \theta}{\partial \xi}\bigg|_{\Xi^-}, \tag{10}$$

where κ is $+1$ for the melting problem or k_S/k_L for the solidification problem. On Eq. (10), the plus sign on the LHS refers to the melting problem, whilst the minus sign refers to the solidification problem. The interface and initial conditions are

$$\theta(\Xi(\tau), \tau) = 0, \tag{11}$$

$$\theta(\xi, 0) = 0, \quad \xi \geqslant 0, \tag{12}$$

$$\Xi(0) = 0. \tag{13}$$

The dimensionless boundary heat-power flux becomes

$$\psi''_w(\tau) = \begin{cases} \psi''_p(\tau), & \text{(a)} \\ \theta_\infty(\tau) - \theta_w(\tau), & \text{(b)} \\ \Theta_\infty(\tau) - \Theta_w(\tau), & \text{(c)} \end{cases} \tag{14}$$

where $\Theta = \theta^4 + 4\theta^3 v_m + 6\theta^2 v_m^2 + 4\theta v_m^3$, and $v_m = T_m/T^\star$.

Stefan problems with imposed heat-power flux admit exact, explicit solutions in Cartesian coordinates only when the heat-power flux imposed at the boundary varies as $\propto t^{-1/2}$ [14], which was shown by Tarzia [15] to be equivalent to an imposed temperature boundary condition; hence, to the classical one-phase Stefan problem. If the latent heat L is made a linear function of the distance x, there is an exact solution due to Voller et al. [16] for a *constant* heat-power flux imposed at the boundary. Voller et al.'s model is representative to the movement of the shoreline in a sedimentary basin.

A large Stefan number approximate analytical solution due to Gammon and Howarth [17] for freezing of liquid spheres was obtained for *constant* temperature and heat-power fluxes using a perturbation method and slight axisymmetric perturbations of the boundary conditions.

An asymptotic analysis of the problem with convective boundary condition for $t \to \infty$ was made by Tarzia and Turner [7]. Howarth [8] presented a perturbation solution for large Ste and $t \to 0, \infty$. The existence, uniqueness, stability, monotone dependence, and asymptotic behaviour for $t \to \infty$ of the problem with prescribed heat-power flux was studied by Cannon and Primicerio [18].

3. Asymptotic solution for $Bi \to 0$

In the limit of $Bi \to 0$ the governing equation (9) reduces to

$$\frac{\partial^2 \theta}{\partial \xi^2} = 0, \; 0 \leqslant \xi < \Xi(\tau), \; \to \; \frac{\partial \theta}{\partial \xi} = f(\tau). \tag{15}$$

Further, the boundary condition, Eq. (8), yields

$$\left.\frac{\partial\theta}{\partial\xi}\right|_0 = \frac{\partial\theta}{\partial\xi} = 0 \;\rightharpoondown\; \theta(\xi,\tau):\theta(\tau), \tag{16}$$

from Eq. (11), one has

$$\theta(\xi,\tau) = 0, \tag{17}$$

leading to an isothermal domain $\xi \geqslant 0$ at the phase change temperature during the whole duration of the problem, yielding $\theta_w = \Theta_w = 0$, in Eq. (14).

The indetermination on the Stefan condition can be resolved by rewriting it in terms of the boundary condition, Eq. (8), using the solution to the governing equation (15):

$$\left.\frac{-\kappa}{Bi}\frac{\partial\theta}{\partial\xi}\right|_{\Xi^-} = \left.\frac{-\kappa}{Bi}\frac{\partial\theta}{\partial\xi}\right|_0, \;\rightharpoondown\; \pm\frac{d\Xi}{d\tau} = \psi''_w(\tau). \tag{18}$$

Melting problems have $\psi''_w(\tau) \geqslant 0$, whilst solidification ones have $\psi''_w(\tau) \leqslant 0$, thus

$$\frac{d\Xi}{d\tau} = |\psi''_w(\tau)|. \tag{19}$$

The solution to Eq. (19) subjected to the initial condition (13) is easily obtained:

$$\Xi(\tau) = \int_0^\tau |\psi''_w(s)|\,ds. \tag{20}$$

3.1. Interpretation

The limit $Bi \to 0$ of the one-phase Stefan problem can be regarded as the "zero-phase Stefan problem", since the heat diffusion equation does not need to be solved for any of the two phases present in the domain. The Stefan condition is reduced to a statement of conservation of energy, Eq. (19). A trivial solution is obtained for the heat diffusion equation on both phases; nonetheless, the Stefan condition admits a non-trivial solution, imparting theoretical and practical significance to the limiting case, in terms of interface location and melting/solidification times [19].

The limit of $Bi \to 0$ corresponds to the phase change equivalent of the lumped capacitance model for problems not involving change of phase.

An interesting outcome is the linearity and simplicity of Eq. (19), by which the dimensionless phase change interface speed equals the absolute dimensionless heat-power flux at the boundary at all times, regardless of the domain thickness $\Xi(\tau)$. Moreover, Eq. (19) shows that phase change interfaces only move away from the fixed boundary.

Eqs. (17) and (20) are the exact solutions to the problem in the limit of $Bi \to 0$ and to problems where the heat-power flux is at all times applied directly at the phase change interface, since for the zero-phase model, the heat-power flux is uniform in the domain $0 \leqslant \xi < \Xi(\tau)$, Eq. (15).

The zero-phase model can also be regarded as an analytical approximation for real phase change processes with small but finite Bi. As for any other analytical approximation of a phase change problem, there is no available way of estimating the error introduced by the simplifications [2].

If the Biot number, Eq. (7), approaches zero due to a very large k_L while keeping all other scales finite, an isothermal condition will appear for the solidification problem only if $\kappa = k_S/k_L$ remains finite, i.e., only if k_S is very large as well.

3.2. Comparison with Leĭbenzon's approximation

The simplification introduced by the zero-phase model is even greater than the one proposed in 1931 by Leĭbenzon [3], which is better known as the "quasistationary approximation", for which a steady heat diffusion equation has to be solved for one of the phases.

Leĭbenzon's widely employed model corresponds to the limit of $Ste \to 0$ by making $c_{L,S} \to 0$ and keeping the temperature or heat-power flux scale at the boundary finite. Approximate solutions of phase change problems under Leĭbenzon's assumption are known for unsteady imposed temperature, convective, and prescribed heat-power flux boundary conditions in Cartesian, cylindrical and spherical coordinates. In the case of steady convective boundary condition, Leĭbenzon's method yields $\Xi(\tau)$ growing as the square-root of time for $Bi \to \infty$, and as a linear function of time as Bi decreases towards zero [2]:

$$\Xi^{qs}(\tau) = \frac{1}{Bi}\left\{\sqrt{1 + 2Bi^2\int_0^\tau \theta_\infty(s)\,ds} - 1\right\}, \quad \text{therefore,} \tag{21}$$

$$\lim_{Bi\to 0}\Xi^{qs}(\tau) = Bi\int_0^\tau \theta_\infty(s)\,ds. \tag{22}$$

Although $\Xi^{qs}(\tau)$ clearly vanishes in Eq. (22), since both parameters of the problem tend to zero, it behaves linearly in time for small values of Bi and constant $\theta_\infty(\tau)$.

3.3. Comparison with Neumann's solution to the classical one-phase problem

The constant temperature boundary condition imposed at $x = 0$ for the classical one-phase Stefan problem was shown by Tarzia [15] to be equivalent to an imposed heat-power flux varying as $q''_p \propto t^{-1/2}$, allowing the comparison of freezing constants

$$\lambda = \frac{X(t)}{2\sqrt{\alpha_{L,S}t}}, \quad \alpha_{L,S} = \frac{k_{L,S}}{\rho c_{L,S}}, \tag{23}$$

given by the Neumann's exact solution to the one-phase problem, and the one given by the present zero-phase model, λ^{zp}.

Let Q be the ratio of heat-power fluxes at the fixed and moving boundaries, according to the Neumann's solution, one has,

$$Q = \frac{q''(0,t)}{q''(X,t)} = e^{\lambda^2}, \quad \therefore \quad Q \geqslant 1. \tag{24}$$

Since Eq. (15) results in an uniform heat-power flux throughout the domain $0 \leqslant \xi \leqslant \Xi(\tau)$ for the zero-phase model, one has $Q^{zp} = 1$; therefore, the model predicts a phase change interface that goes Q times faster than the value given by Neumann's solution by virtue of the Stefan condition, Eq. (10), yielding

$$\lambda^{zp} = Q\lambda = \lambda e^{\lambda^2} = \frac{Ste}{\sqrt{\pi}\,\mathrm{erf}\,\lambda}, \quad \therefore \quad \lambda^{zp} \geqslant \lambda. \tag{25}$$

Hence, the zero-phase model overestimates the phase change interface position for $Bi > 0$. This is also true for the Leĭbenzon's analytical approximation [2].

4. Relaxation of constraints

Knowledge of the zero-phase (isothermal) condition obtained once the Biot number goes to zero allows the relaxation of some of the original assumptions, making the model applicable to more realistic situations.

4.1. Arbitrary interface initial position

The most elementary relaxation to the problem expressed by Eqs. (8)–(11), (13), and (14) is to change the initial location of the liquid–solid interface, Eq. (13), to $\Xi(0) = \Xi_0$, $\Xi_0 \geqslant 0$. Evidently, Eq. (12) applies now to both phases. The zero-phase model solution is simply

$$\Xi(\tau) = \Xi_0 + \int_0^\tau |\psi''_w(s)|\,ds. \tag{26}$$

It is worth noting that the initial condition has to be of an uniform temperature $\theta(x, 0) = 0$, that will last while phase change takes place.

4.2. Presence of multiple interfaces on the domain

Now a number $J > 1$ of interfaces $\Xi_{j,0} \geqslant 0$, $0 \leqslant j \leqslant (J-1)$, are initially present in the domain. The domain is therefore comprised initially of planar, parallel regions occupied by solid and liquid phase change material. Still assuming no sign switch in the boundary heat-power flux, the initial location of the first (leftmost) interface for a melting problem is zero if solid occupies initially the region adjacent to the boundary, as in Fig. 1(a), or greater than zero otherwise, as in Fig. 1(b), or the inverse, for a freezing problem. For $\tau = 0$, the position of the jth interface is denoted as $\Xi_{j,0}$ and for $\tau \geqslant 0$, as $\Xi_j(\tau)$, such that $\Xi_j(0) \equiv \Xi_{j,0}$.

The Stefan condition, Eq. (10), states that all incoming flux of energy is entirely absorbed or released by the material in the form of latent heat at the first interface. Therefore, the heat-power flux on the back side of the first interface and locations beyond that is always zero, due to the absence of temperature gradients and unbounded domain; hence, the first interface moves forward according to the solution of Eq. (19), while all other interfaces remain static.

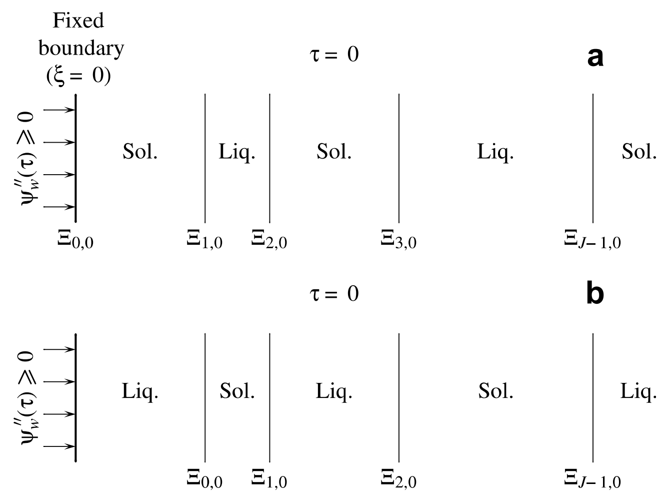

Fig. 1. Domain initially with multiple interfaces. (a) Solid as first phase with $J = 5$. (b) Liquid as first phase with $J = 4$.

The first region occupied by solid PCM is $\Xi_0(\tau) \leqslant \xi \leqslant \Xi_{1,0}$, for a melting problem. When $\Xi_0(\tau)$ reaches $\Xi_{1,0}$, the first region occupied by solid, along with the Ξ_0 and Ξ_1 interfaces that define it, disappear and the interface Ξ_2, if present, instantly becomes the moving one.

It is worth noting that in the zero-phase model, conservation of energy is satisfied *only* by phase change, and not at all by the governing equation, in the form of sensible heat; therefore, the model will yield physically meaningful results only if there is at least one phase change interface in the domain.

Let dimensionless times $\tau_i^\star$, with $\tau_0^\star = 0$ and $\tau_i^\star$, $i > 0$ being the dimensionless times for which $\Xi_{2i-2}(\tau_i^\star) = \Xi_{2i-1,0}$.

Starting with $i = 1$, the time $\tau_i^\star$ is the solution of

$$\int_{\tau_{i-1}^\star}^{\tau_i^\star} |\psi''_w(s)|\,ds = \Xi_{2i-1,0} - \Xi_{2i-2,0}, \tag{27}$$

or infinity, if there is no interface $\Xi_{2i-1,0}$. There will be as many $\tau_i^\star, i \geqslant 1$, as regions initially solid for a melting problem.

The position of the moving, leftmost interface $\Xi_{2i}(\tau)$, $i \geqslant 0$ is

$$\Xi_{2i}(\tau) = \Xi_{2i,0} + \int_{\tau_i^\star}^{\tau} |\psi''_w(s)|\,ds, \quad \tau_i^\star \leqslant \tau < \tau_{i+1}^\star. \tag{28}$$

At all times $\tau_i^\star$, $i \geqslant 1$, the moving interface jumps a dimensionless length of $\Xi_{2i,0} - \Xi_{2i-1,0}$.

4.3. Sign-switching boundary heat-power flux

Though Leĭbenzon's quasistationary approximation is widely used as a tool to provide simple, order of magnitude estimates of the basic behaviour of real systems, its inability of matching certain types of initial conditions prevents its use in cases where the direction of heat transfer changes at the boundary. This kind of situation, howbeit, seems to represent the most common application of PCMs undergoing cyclic charge and discharge of thermal energy in practice [20,21].

Solutions for zero-phase problems with sign-switching heat-power fluxes at the boundary can be found, since the whole domain remains isothermal at all times, provided phase change is always taking place.

If an initially solid domain at the melting temperature is melted under the influence of an initially positive heat-power flux imposed at its boundary, the melting interface will depart from the fixed boundary and move away from it, satisfying the Stefan condition. If the heat-power flux imposed at the boundary reverses sign, the melting interface cannot move backwards, Eq. (19); instead, for $Bi \to 0$ a solidifying front instantly appears at the fixed boundary, where heat is being withdrawn. The solidifying front will move away from the boundary as well. Therefore, in zero-phase modeling, interfaces can only move away from the excitation at a fixed boundary, and a reversal of sign in the boundary thermal excitation always introduces new interfaces.

Let Σ_0^L and Σ_0^S be the initial dimensionless extent of liquid and solid phases, respectively. On Fig. 1(a), for instance, $\Sigma_0^L = (\Xi_{2,0} - \Xi_{1,0}) + (\Xi_{4,0} - \Xi_{3,0})$, while $\Sigma_0^S = \infty$. Phase change will always occur if the following condition holds true:

$$-\Sigma_0^L \leqslant \int_0^\tau \psi''_{\mathrm{w}}(s)\,\mathrm{d}s \leqslant \Sigma_0^S, \quad \forall\tau \geqslant 0. \tag{29}$$

Eqs. (27) and (28) are the solutions of the problem until the heat-power flux imposed at the boundary changes sign. Whenever $\psi''_{\mathrm{w}}(\tau)$ changes sign, say at dimensionless times $\tau = \tau_k, k = 1, 2, 3, \ldots,$ one have to treat the problem as one of multiple interfaces in terms of a translated dimensionless time variable $\tau' = \tau - \tau_k$ with a new leftmost moving interface $\Xi_{0,0}$ set to zero, in accordance to what was conventioned in Section 4.2. The remaining interfaces have to be re-indexed to fit in the new sequence.

4.4. Finite domain

This assumption renders the model akin to real applications, in which multiple interfaces are allowed to coexist inside the domain of length $x^\star\ell$. In this configuration, another boundary condition ought to be specified on the rightmost end of the domain, $\xi = \ell$.

Assuming that a positive heat-power flux is oriented so to *enter* the left boundary $\xi = 0$, and to *leave* through the right boundary, $\xi = \ell$. Let $\psi''_\ell(\tau)$ and $\psi''_{\mathrm{r}}(\tau)$ be the dimensionless heat-power fluxes in the form of Eq. (14) at the left and right boundaries, respectively. Fig. 2 illustrates an initial condition for which both ψ''_ℓ and ψ''_{r} are positive.

Phase changes will always occur if Eq. (30) holds true:

$$-\Sigma_0^L \leqslant \int_0^\tau [\psi''_\ell(s) - \psi''_{\mathrm{r}}(s)]\,\mathrm{d}s \leqslant \ell - \Sigma_0^L, \quad \forall\tau \geqslant 0. \tag{30}$$

If two or more phase change interfaces are present in the domain, there will be no heat interaction between left

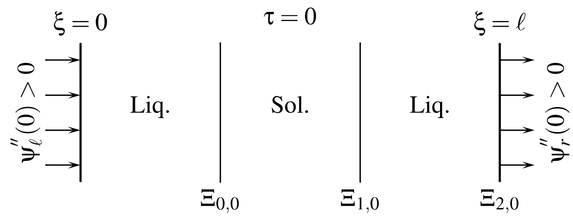

Fig. 2. Finite domain with multiple interfaces.

and right sides of the domain, and only the left and rightmost interfaces will move in response to the heat exchange at their respective boundaries. While this condition holds true, the problem can be separated into two independent ones, and the solutions can be obtained from the methods already discussed in Sections 4.1–4.3 with the Stefan condition for the leftmost and rightmost interfaces being, respectively,

$$\frac{\mathrm{d}\Xi_\ell}{\mathrm{d}\tau} = |\psi''_\ell(\tau)|, \quad \text{and} \quad \frac{\mathrm{d}\Xi_{\mathrm{r}}}{\mathrm{d}\tau} = -|\psi''_{\mathrm{r}}(\tau)|. \tag{31}$$

If, however, there is only one interface present in the domain, then the Stefan condition becomes

$$\frac{\mathrm{d}\Xi}{\mathrm{d}\tau} = |\psi''_\ell(\tau)| - |\psi''_{\mathrm{r}}(\tau)|. \tag{32}$$

It is worth noting that this condition can only be achieved if ψ''_ℓ and ψ''_{r} have the same sign, for an odd number of interfaces must be present in the domain if their signs agree or an even number otherwise. The solution is therefore

$$\Xi(\tau) = \Xi_0 + \int_0^\tau [|\psi''_\ell(s)| - |\psi''_{\mathrm{r}}(s)|]\,\mathrm{d}s \tag{33}$$

until either $\psi''_\ell(\tau)$ or $\psi''_{\mathrm{r}}(\tau)$ or both changes sign.

4.5. Cylindrical coordinates

Let r be the radial cylindrical coordinate and $R_j(t)$ the interfaces' radial locations. Using the scales employed in the Cartesian coordinates problem with the new dimensionless variables

$$(\zeta, Z) = \frac{(r, R)}{x^\star}, \tag{34}$$

as shown in Fig. 3, the dimensionless problem in cylindrical coordinates is [22,23]:

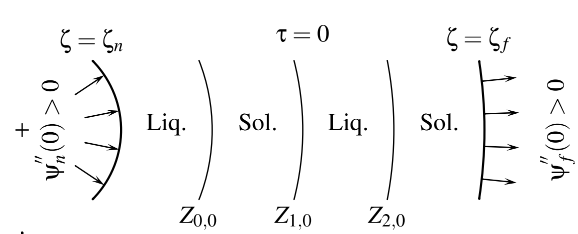

Fig. 3. Finite hollow cylinder with multiple interfaces and $J = 3$.

$$\psi''_{\mathrm{n}}(\tau) = \frac{-\kappa_{\mathrm{n}}}{Bi}\frac{\partial\theta}{\partial\zeta}\bigg|_{\zeta_{\mathrm{n}}}, \tag{35}$$

$$\psi''_{\mathrm{f}}(\tau) = \frac{-\kappa_{\mathrm{f}}}{Bi}\frac{\partial\theta}{\partial\zeta}\bigg|_{\zeta_{\mathrm{f}}}, \tag{36}$$

$$Ste\frac{\partial\theta}{\partial\tau} = \frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\frac{\kappa_j}{Bi}\zeta\frac{\partial\theta}{\partial\zeta}\right), \quad Z_j(\tau) < \zeta < Z_{j+1}(\tau), \tag{37}$$

$$\pm\frac{\mathrm{d}Z_j}{\mathrm{d}\tau} = \frac{-\kappa}{Bi}\frac{\partial\theta}{\partial\zeta}\bigg|_{Z_j^-} + \frac{\kappa}{Bi}\frac{\partial\theta}{\partial\zeta}\bigg|_{Z_j^+}, \tag{38}$$

$$\theta\big(Z_j(\tau),\tau\big) = 0, \tag{39}$$

$$\theta(\zeta,0) = 0, \quad \zeta_{\mathrm{n}} \leqslant \zeta \leqslant \zeta_{\mathrm{f}}, \tag{40}$$

$$Z_j(0) = Z_{j,0}. \tag{41}$$

In Eqs. (35)–(41), κ is +1 if the corresponding location or region is occupied by liquid, or $k_{\mathrm{S}}/k_{\mathrm{L}}$ otherwise. The subscripts n and f refer to the boundaries 'near' and 'far' from the center, as indicated in Fig. 3. On the LHS of Eq. (38), the plus sign applies to interfaces where liquid PCM occupies their Z_j^- side, while the minus sign applies to the opposite configuration. It is worth noting that κ on the first term of RHS of Eq. (38) will always be different than the one on the second term, since each side of the interface is occupied by a different phase.

The asymptotic behaviour for $Bi \to 0$ yields, for Eq. (37)

$$\left(\zeta\frac{\partial\theta}{\partial\zeta}\right) = f(\tau). \tag{42}$$

It is worth comparing the influence of geometry on Eqs. (42) and (15). The indetermination on the Stefan condition is dealt with in a similar manner as done in the Cartesian coordinates case. Three situations arise in the solution of the Stefan condition: (i) zero heat-power flux on the far side of the interface, (ii) zero flux on the near size of the interface, and (iii) non-zero fluxes on both sides of an interface.

Situation (i) arises in cases of outward phase change, where either the cylinder is radially unbounded from the outside (infinite cylinder), or when there are other interfaces between the innermost one and the outer boundary, as is the case of interface Z_0 in Fig. 3. The Stefan condition for the innermost interface becomes, after eliminating the indetermination, $Z\mathrm{d}Z/\mathrm{d}\tau = \zeta_{\mathrm{n}}|\psi''_{\mathrm{n}}(\tau)|$. The solution is

$$Z_0(\tau) = \sqrt{Z_{0,0}^2 + 2\int_0^\tau \zeta_{\mathrm{n}}|\psi''_{\mathrm{n}}(s)|\,\mathrm{d}s}. \tag{43}$$

Situation (ii) arises in cases of inward phase change, where either the cylinder is unbounded from the inside (filled cylinder), or when there are other interfaces between the outermost one and the inner boundary, as is the case of interface Z_2 in Fig. 3. The Stefan condition for the outermost interface becomes, after eliminating the indetermination, $Z\mathrm{d}Z/\mathrm{d}\tau = -\zeta_{\mathrm{f}}|\psi''_{\mathrm{f}}(\tau)|$. The solution is

$$Z_{J-1}(\tau) = \sqrt{Z_{J-1,0}^2 - 2\int_0^\tau \zeta_{\mathrm{f}}|\psi''_{\mathrm{f}}(s)|\,\mathrm{d}s}. \tag{44}$$

Situation (iii) only arises in radially finite hollow cylinders having only one phase change interface. As mentioned on Section 4.4, this condition can only be achieved if ψ''_{n} and ψ''_{f} have the same sign. The Stefan condition for the only interface becomes, after eliminating the indetermination, $Z\mathrm{d}Z/\mathrm{d}\tau = \zeta_{\mathrm{n}}|\psi''_{\mathrm{n}}(\tau)| - \zeta_{\mathrm{f}}|\psi''_{\mathrm{f}}(\tau)|$. The solution is

$$Z(\tau) = \sqrt{Z_0^2 + 2\int_0^\tau [\zeta_{\mathrm{n}}|\psi''_{\mathrm{n}}(s)| - \zeta_{\mathrm{f}}|\psi''_{\mathrm{f}}(s)|]\,\mathrm{d}s} \tag{45}$$

until either $\psi''_{\mathrm{n}}(\tau)$ or $\psi''_{\mathrm{f}}(\tau)$ or both changes sign. Analogous methods to those presented in Sections 4.2 and 4.3 should be used in case multiple interfaces are present and sign-switching in the heat-power flux occurs for situations (i), (ii), and (iii).

4.6. Spherical coordinates

Using the same scaling as in cylindrical coordinates, the dimensionless problem in spherical coordinates [22,23] is similar to the one in cylindrical coordinates, Eqs. (35)–(41), with the exception of the governing equation:

$$Ste\frac{\partial\theta}{\partial\tau} = \frac{1}{\zeta^2}\frac{\partial}{\partial\zeta}\left(\frac{\kappa_j}{Bi}\zeta^2\frac{\partial\theta}{\partial\zeta}\right), \quad Z_j(\tau) < \zeta < Z_{j+1}(\tau), \tag{46}$$

The asymptotic behaviour for $Bi \to 0$ yields

$$\left(\zeta^2\frac{\partial\theta}{\partial\zeta}\right) = f(\tau). \tag{47}$$

The effect of geometry is evident by comparing Eqs. (47), (42) and (15). Similar situations arise in the solution of the Stefan condition as in cylindrical coordinates:

Outward phase change (i), where the Stefan condition reads $Z^2\mathrm{d}Z/\mathrm{d}\tau = \zeta_{\mathrm{n}}^2|\psi''_{\mathrm{n}}(\tau)|$, and admits the solution

$$Z_0(\tau) = \sqrt[3]{Z_{0,0}^3 + 3\int_0^\tau \zeta_{\mathrm{n}}^2|\psi''_{\mathrm{n}}(s)|\,\mathrm{d}s}. \tag{48}$$

Inward phase change (ii), where the Stefan condition reads $Z^2\mathrm{d}Z/\mathrm{d}\tau = -\zeta_{\mathrm{f}}^2|\psi''_{\mathrm{f}}(\tau)|$, and admits the solution

$$Z_{J-1}(\tau) = \sqrt[3]{Z_{J-1,0}^3 - 3\int_0^\tau \zeta_{\mathrm{f}}^2|\psi''_{\mathrm{f}}(s)|\,\mathrm{d}s}. \tag{49}$$

And finite hollow sphere (iii), where the Stefan condition reads $Z^2\mathrm{d}Z/\mathrm{d}\tau = \zeta_{\mathrm{n}}^2|\psi''_{\mathrm{n}}(\tau)| - \zeta_{\mathrm{f}}^2|\psi''_{\mathrm{f}}(\tau)|$, and admits the solution

$$Z(\tau) = \sqrt[3]{Z_0^3 + 3\int_0^\tau [\zeta_{\mathrm{n}}^2|\psi''_{\mathrm{n}}(s)| - \zeta_{\mathrm{f}}^2|\psi''_{\mathrm{f}}(s)|]\,\mathrm{d}s} \tag{50}$$

until either $\psi''_{\mathrm{n}}(\tau)$ or $\psi''_{\mathrm{f}}(\tau)$ or both changes sign. Similar methods to those presented in Sections 4.2 and 4.3 should be used in case multiple interfaces are present and sign-switching in the heat-power flux occurs for situations (i), (ii), and (iii).

5. Generalization

Solutions (33), (45) and (50) exhibit a pattern that allows one to write a more general solution for the Stefan condition in Cartesian, cylindrical, and spherical coordinates for melting and solidification problems. The physical interpretation of such solution is a statement of the principle of conservation of energy (from which the Stefan condition itself is derived) subjected to the geometrical configuration of the problem and the sense of phase change taking place, neglecting the sensible heat, due to the asymptotic behaviour $Bi \to 0$; hence, for an interface $Z_j(\tau)$,

$$Z_j(\tau) = \left\{ Z_{j,0}^G + G \int_0^\tau [\zeta_{\rm n}^{(G-1)} |\psi_{\rm n}''(s)| - \zeta_{\rm f}^{(G-1)} |\psi_{\rm f}''(s)|] \, {\rm d}s \right\}^{\frac{1}{G}}, \tag{51}$$

where G is a geometrical coefficient: $G = 1$ for Cartesian coordinates, $(\xi, \Xi) \equiv (\zeta, Z)$; $G = 2$ for cylindrical coordinates, and $G = 3$ for spherical coordinates, as the asymptotic behaviour of the governing equation for $Bi \to 0$ requires

$$\zeta^{(G-1)} \frac{\partial \theta}{\partial \zeta} = f(\tau), \tag{52}$$

and the Stefan condition becomes, after eliminating the indetermination,

$$Z^{G-1} \frac{{\rm d}Z}{{\rm d}\tau} = \zeta_{\rm n}^{G-1} |\psi_{\rm n}''(\tau)| - \zeta_{\rm f}^{G-1} |\psi_{\rm f}''(\tau)|. \tag{53}$$

6. Conclusions

Starting from the classical one-phase Stefan problem with the boundary condition changed to specified heat-power flux, it was shown that an isothermal condition characterizes the asymptotic behaviour of the system for $Bi \to 0$, yielding thus a moving boundary problem in which the heat diffusion equation does not need to be solved for any phase; hence, the zero-phase Stefan problem.

In this zero-phase model, only the ODE for the position of the moving boundary, which is obtained by eliminating the indetermination in the Stefan condition by imposing conservation of energy, is left to be solved, admitting a very simple solution.

Solution procedures for cases involving multiple interfaces and sign-switching heat-power flux boundary conditions were outlined for cases in Cartesian coordinates. Analogous procedures are seemingly easy to follow for other coordinates.

The simple, lumped nature of the zero-phase problem allows explicit solutions to be obtained even when several restrictions of the original problem are relaxed. Thus, solutions for the zero-phase Stefan problem are found for finite unidimensional systems in Cartesian, cylindrical, and spherical coordinates and for time-varying, sign-switching boundary conditions and multiphase however isothermal initial conditions. All the aforementioned solutions share common features, allowing for a general expression (51) to be obtained, in which the type of geometry is represented by an integer parameter.

The asymptotic solutions derived in the present work can be used as analytical approximation for cases where the Biot number is small but finite. Such a simple phase change model can be employed as an approximate tool in a great number of practical situations; however, it is only able to estimate the position of the phase change interface(s), or, inversely, the melting and/or freezing times.

The zero-phase problem is a lower order of approximation than the widely used Leĭbenzon's quasistationary method, as far as the ability of having a temperature profile in one of the phases is concerned. This feature can be viewed either as an over-simplification, or the further removal of a burden from Leĭbenzon's method that allows the present treatment to approach problems with sign-switching boundary conditions.

The zero-phase problem shares important common features with Leĭbenzon's method, viz., (i) the heat-power flux on a phase adjacent to a boundary is uniform, and identical to that imposed on the boundary; (ii) changes on the heat-power flux at the boundary are instantly felt on the interface, producing an instant away effect at a distance; (iii) neglects sensible heat effects; (iv) overestimates the interface position when applied as an analytical approximation; (v) have no error estimates when used as an approximation [2]; and (vi) admits convective and imposed flux boundary conditions.

However, contrary to Leĭbenzon's method, the zero-phase problem (i) can only mimic imposed temperature boundary conditions by making $q''_{\rm p} \propto t^{-1/2}$, even though the phase change domain will remain isothermal; (ii) cannot satisfy an initially linear temperature profile, not fully representing an one- or two-phase problem.

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